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Bicliques and Eigenvalues

Willem H. Haemers

Department of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands
E-mail: haemers@kub.nl

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A biclique in a graph Γ is a complete bipartite subgraph of Γ. We give bounds for the maximum number of edges in a biclique in terms of the eigenvalues of matrix representations of Γ. These bounds show a strong similarity with Lovász's

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1. INTRODUCTION

Throughout $\Gamma=(V, E)$ is a graph with v vertices. A pair of two disjoint subsets A and B of V is a *biclique* if $\{a, b\} \in E$ for all $a \in A$ and $b \in B$. Thus the edges $\{a, b\}$ form a complete bipartite subgraph of Γ (which is in general not an induced subgraph). A biclique $\{A, B\}$ clearly has $|A| \cdot |B|$ edges. We define $\Phi(\Gamma)$ to be the maximum of $\sqrt{|A| \cdot |B|}$ for bicliques $\{A, B\}$ in Γ.

In $\bar{\Gamma}$, the complement of Γ, a biclique $\{A, B\}$ becomes a pair of disjoint subsets of vertices with no edges between them. Such subsets are called *disconnected vertex sets*. In an earlier paper [4] we obtained an upper bound for $\Phi(\Gamma)$ (in terms of disconnected vertex sets). There the bound was applied to equidistant code pairs, using the theory of association schemes. It turned out that, at least for these applications, the bound is rather good. In many cases it is tight or almost tight. In the present paper we focus on the similarity of this bound with Lovász's bound $\vartheta(\Gamma)$ for the maximum size $\omega(\Gamma)$ of a clique.[1] To do so we use a variation on $\vartheta(\Gamma)$, and introduce a slightly weaker bound $\vartheta'(\Gamma)$ for $\omega(\Gamma)$. Like $\vartheta(\Gamma)$, the bound $\vartheta'(\Gamma)$ behaves nicely with respect to a certain graph product which makes it into a bound for the so called Shannon capacity (see the next section). Analogously we derive bounds $\varphi(\Gamma)$ and $\varphi'(\Gamma)$ for $\Phi(\Gamma)$. Here the second bound $\varphi'(\Gamma)$ behaves nicely with respect to the graph product. This makes

[1] We follow the definition of $\vartheta(\Gamma)$ from [8], which corresponds to $\vartheta(\bar{\Gamma})$ in [7].

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$\varphi'(\Gamma)$ an upper bound for $\Phi'(\Gamma)$, the bicapacity of Γ, a biclique analogue of the Shannon capacity that we introduce in Section 2. Finally we discuss some aspects of the computation of $\varphi(\Gamma)$, $\varphi'(\Gamma)$, $\vartheta'(\Gamma)$, and $\Phi(\Gamma)$.

We introduce some matrix notation. The all-one matrix is denoted by J and the all-one vector by $\underline{1}$. The eigenvalues of a symmetric $v \times v$ matrix A are denoted by $\lambda_1(A)$, ..., $\lambda_v(A)$ and we define

$$\lambda_{\max}(A) = \max_i \{\lambda_i(A)\}, \qquad \lambda_{\min}(A) = \min_i \{\lambda_i(A)\}, \qquad \text{and}$$

$$\lambda_{\text{abs}}(A) = \max_i \{|\lambda_i(A)|\}.$$

If A has constant row (and column) sums, k say, then k is an eigenvalue. We put $\lambda_1(A) = k$ and define

$$\lambda_{\max'}(A) = \max_{i \neq 1} \{\lambda_i(A)\}, \qquad \lambda_{\min'}(A) = \min_{i \neq 1} \{\lambda_i(A)\}, \qquad \text{and}$$

$$\lambda_{\text{abs}'}(A) = \max_{i \neq 1} \{|\lambda_i(A)|\}.$$

Note that if k is a multiple eigenvalue then $\lambda_{\max'}(A) = \lambda_{\max}(A)$, etc.. Suppose B is another symmetric matrix. We recall that $\lambda_{i+v(j-1)}(A \otimes B) = \lambda_i(A)\, \lambda_j(B)$, where $A \otimes B$ denotes the Kronecker (or tensor) product of matrices A and B.

2. GRAPH PRODUCTS

Consider two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$. The product $\Gamma_1 * \Gamma_2$ is the graph with vertex set $V_1 \times V_2$, where two vertices (x_1, x_2) and (y_1, y_2) are adjacent whenever $\{x_1, y_1\} \in E_1$ or $\{x_2, y_2\} \in E_2$.

Suppose $C_1 \subset V_1$ and $C_2 \subset V_2$ are cliques in Γ_1 and Γ_2, respectively. Then clearly $C_1 \times C_2$ is a clique in $\Gamma_1 * \Gamma_2$. So

$$\omega(\Gamma_1 * \Gamma_2) \geqslant \omega(\Gamma_1)\, \omega(\Gamma_2).$$

Let Γ^k denote the product of k copies of Γ. Then $\omega(\Gamma^k) \geqslant (\omega(\Gamma))^k$. The *Shannon capacity* $\Theta(\Gamma)$ is defined by

$$\Theta(\Gamma) = \sup_k \sqrt[k]{\omega(\Gamma^k)}.$$

The quantity $\Theta(\Gamma)$, or more precisely $\log \Theta(\Gamma)$,is a concept from information theory which has been introduced by Shannon [11]. It is clear that

$\Theta(\Gamma) \geqslant \omega(\Gamma)$. For many graphs the strict inequality holds. The pentagon C_5, for example, has $\omega(C_5^2) = 5$ so

$$\Theta(C_5) \geqslant \sqrt{5} > \omega(C_5) = 2.$$

In fact, Lovász's upper bound $\vartheta(\Gamma)$ [7] for $\Theta(\Gamma)$ equals $\sqrt{5}$ for C_5. So $\Theta(C_5) = \sqrt{5}$. We shall give a slightly different proof of this in the next section.

Next suppose that $\{A_1, B_1\}$ and $\{A_2, B_2\}$ are bicliques in Γ_1 and Γ_2, respectively. It is straightforward that $\{A_1 \times V_2, B_1 \times V_2\}$ and $\{A_2 \times V_1, B_2 \times V_1\}$ are bicliques in $\Gamma_1 * \Gamma_2$. Hence

$$\Phi(\Gamma_1 * \Gamma_2) \geqslant \max\{v_2 \Phi(\Gamma_1), v_1 \Phi(\Gamma_2)\}$$

(where $v_1 = |V_1|$ and $v_2 = |V_2|$) and also $\Phi(\Gamma^k) \geqslant v^{k-1} \Phi(\Gamma)$. Define

$$\Phi'(\Gamma) = \sup_k \Phi(\Gamma^k)/v^{k-1}.$$

The supremum exists, since $\Phi(\Gamma^k) \leqslant v^k$. We call $\Phi'(\Gamma)$ the *bicapacity* of Γ (although we know of no application to information theory). It is clear that $\Phi'(\Gamma) \geqslant \Phi(\Gamma)$. Also here strict inequality can hold. Indeed, the complete graph K_v with v odd satisfies $\Phi(K_v) = \frac{1}{2}\sqrt{v^2 - 1}$ and $K_v^k = K_{v^k}$, so

$$\Phi'(K_v) = \sup_k \frac{\sqrt{v^{2k} - 1}}{2v^{k-1}} = \frac{v}{2} > \Phi(K_v) = \frac{\sqrt{v^2 - 1}}{2}.$$

In Section 4 bounds for $\Phi(\Gamma)$ and $\Phi'(\Gamma)$ will be derived together with some less trivial examples with $\Phi'(\Gamma) > \Phi(\Gamma)$.

3. A VARIATION ON LOVASZ'S BOUND

Let $\mathcal{N}_\Gamma$ denote the class of symmetric matrices N indexed by the vertices of Γ with the property that $(N)_{i,j} = 1$ if $\{i, j\} \in E$ or $i = j$. Lovász's $\vartheta(\Gamma)$ [7] can be defined as (see [8, p. 77])

$$\vartheta(\Gamma) = \min_{N \in \mathcal{N}_\Gamma} \lambda_{\max}(N).$$

PROPOSITION 3.1. $\omega(\Gamma) \leqslant \vartheta(\Gamma)$.

Proof. A clique of size $\omega = \omega(\Gamma)$ corresponds to a $\omega \times \omega$ submatrix J in N. We have $\lambda_{\max}(J) = \omega$, and eigenvalue interlacing gives $\lambda_{\max}(J) \leqslant \lambda_{\max}(N)$. ∎

Lovász proved that $\vartheta(\Gamma_1 * \Gamma_2) \leqslant \vartheta(\Gamma_1)\,\vartheta(\Gamma_2)$, which implies that $\omega(\Gamma^k) \leqslant \vartheta(\Gamma^k) \leqslant \vartheta^k(\Gamma)$, so $\Theta(\Gamma) \leqslant \vartheta(\Gamma)$. We shall do something similar with $\vartheta'(\Gamma)$, which is defined as follows. Let $\mathcal{N}'_\Gamma$ be the class of symmetric matrices N' indexed by the vertices of Γ with the property that $N'\underline{1} = \underline{1}$ and $(N')_{i,j} = 0$ if $\{i, j\} \in E$ or $i = j$. For some graphs $\mathcal{N}'_\Gamma$ is empty (see Proposition 3.6). If this is the case we put $\vartheta'(\Gamma) = \infty$. For all other graphs we define

$$\vartheta'(\Gamma) = \inf_{N' \in \mathcal{N}'_\Gamma} \frac{v\lambda}{1+\lambda}, \qquad \text{where} \quad \lambda = -\lambda_{\min'}(N').$$

Note that from trace $N' = 0$ and $\lambda_1(N') = 1$ it follows that $\lambda > 0$, so $\vartheta'(\Gamma)$ is well-defined. Moreover, we shall see (Theorem 5.1) that we may restrict ourselves to a bounded subset of $\mathcal{N}'_\Gamma$, so the infimum is actually a minimum. Define $\tilde{\mathcal{N}}'_\Gamma$ be the subset of $\mathcal{N}'_\Gamma$ for which the minimum is achieved.

Proposition 3.2. $\vartheta(\Gamma) \leqslant \vartheta'(\Gamma)$.

Proof. Clearly we may assume that $\vartheta'(\Gamma) < \infty$. Take $N' \in \tilde{\mathcal{N}}'_\Gamma$. Define $N = J - \frac{v}{(1+\lambda)} N'$. Then $N \in \mathcal{N}_\Gamma$, $\lambda_1(N) = v - v/(1+\lambda)$, and $\lambda_i(N) = -v\lambda_i(N')/(1+\lambda)$ for $i \neq 1$. So $\lambda_{\max}(N) = v\lambda/(1+\lambda)$. ∎

Proposition 3.3. $\vartheta'(\Gamma_1 * \Gamma_2) \leqslant \vartheta'(\Gamma_1)\,\vartheta'(\Gamma_2)$.

Proof. Assume that $\vartheta'(\Gamma_1)$ and $\vartheta'(\Gamma_2) < \infty$. Let $N'_1 \in \tilde{\mathcal{N}}'_{\Gamma_1}$ and $N'_2 \in \tilde{\mathcal{N}}'_{\Gamma_2}$ and put $\ell_i = -\lambda_{\min}(N'_i)$. Define $N'_{1,2} = (N'_1 + \ell_1 I) \otimes (N'_2 + \ell_2 I) - \ell_1 \ell_2 I$. Then $N'_{1,2}$ has row sum $k = 1 + \ell_1 + \ell_2$, $\lambda_{\min}(N'_{1,2}) = -\ell_1\ell_2$, and $\frac{1}{k} N'_{1,2} \in \mathcal{N}'_{\Gamma_1 * \Gamma_2}$. Hence

$$\vartheta'(\Gamma_1 * \Gamma_2) \leqslant \frac{v_1 v_2 \ell_1 \ell_2}{k + \ell_1 \ell_2} = \vartheta'(\Gamma_1)\,\vartheta'(\Gamma_2). \quad ∎$$

Proposition 3.3 implies that $\omega(\Gamma^k) \leqslant \vartheta'(\Gamma^k) \leqslant \vartheta'^k(\Gamma)$ and therefore:

Theorem 3.4. $\Theta(\Gamma) \leqslant \vartheta(\Gamma)$.

If Γ is regular of degree k and has adjacency matrix A, then $\frac{1}{k} A \in \mathcal{N}'_{\bar{\Gamma}}$ ($\bar{\Gamma}$ is the complement of Γ). This gives the following result of Lovász [7, Theorem 9].

Corollary 3.5. *Suppose Γ is regular of degree k and let $-\lambda$ be the smallest eigenvalue of the adjacency matrix of Γ. Then*

$$\Theta(\bar{\Gamma}) \leqslant \frac{v\lambda}{k+\lambda}.$$

In particular, the pentagon C_5 has $\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{5}$. Thus we proved $\Theta(C_5) \leqslant \sqrt{5}$, as promised. The above proof for Corollary 3.5 is essentially the same as the one given in [5]. As noted earlier, the set $\mathcal{N}'_\Gamma$ may be empty. The following proposition, which we shall need in Section 5, determines when this is the case.

PROPOSITION 3.6. *Suppose $\bar{\Gamma}$ is a connected graph with at least two vertices.*

(i) *$\mathcal{N}'_\Gamma = \emptyset$ if and only if $\bar{\Gamma}$ is bipartite with parts of unequal size.*

(ii) *If $\mathcal{N}'_\Gamma \neq \emptyset$, then $\mathcal{N}'_\Gamma$ contains a matrix N' such that $|(N')_{i,j}| < v$ $(0 \leqslant i, j \leqslant v)$.*

Proof. If $\bar{\Gamma}$ is bipartite, N' has the form $N' = \left[\begin{smallmatrix} O & A \\ A^\top & O \end{smallmatrix}\right]$, where both A and $A^\top$ have constant row sum 1. This is clearly impossible if A is not a square matrix.

Suppose $\bar{\Gamma} = (V, \bar{E})$ is not of the described form. We shall construct a matrix $N' = (w_{i,j})$ in $\mathcal{N}'_\Gamma$ by assigning a weight w_e to each edge $e \in \bar{E}$, such that the sum of the weights around each vertex equals 1. If $\bar{\Gamma}$ is bipartite (with parts V_1 and V_2 of equal size), take a spanning tree T. Put $w_e = 0$ for all edges e of $\bar{\Gamma}$ that are not in T. Choose a vertex $r \in V_1$ to be the root of T. By starting at the leaves $(\neq r)$ and going down to the root r, we easily find that there is a unique way to assign weights w_e to the edges e of T such that $\sum_{e:\, i \in e} w_e = 1$ for all vertices $i \neq r$. Moreover, $|w_e| < |V|$ for every $e \in \bar{E}$. Define $w(r) = \sum_{e:\, r \in e} w_e$ and observe that $w(r) + |V_1| - 1 = \sum_{e \in \bar{E}} w_e = |V_2|$. Hence $w(r) = 1$.

Next suppose $\bar{\Gamma}$ has a cycle $C = (V_C, E_C)$ of odd length n. Take a spanning forest $F = (V, E_F)$ in $(V, \bar{E} \backslash E_C)$ and put $w_e = 0$ if $e \notin E_F \cup E_C$. For every tree $T = (V_T, E_T)$ of F, choose a root r of T such that $r \in V_C$. For each tree, assign weights to the edges as above and put $w(r) = \sum_{e:\, r \in e \in E_F} w_e$. It follows that $|w(r)| < |V_T|$. Finally we need to determine w_e for each $e \in E_C$. For this we have to solve the following system of linear equations: $w_{\{i,j\}} + w_{\{j,k\}} = 1 - w(j)$, for each incident pair of edges $\{i, j\}$, $\{j, k\}$ in C (take $w(j) = 1$ if j is not a root). The matrix A of this system is circulant$[1, 1, 0, ..., 0]$, and $A^{-1} = \frac{1}{2}$circulant$[1, 1, -1, 1, -1, 1, ..., -1]$ (here we use that n is odd, otherwise A has no inverse). Thus the system has a unique solution and moreover $|w_{\{i,j\}}| \leqslant \frac{1}{2} \sum_{j \in V_C} |1 - w(j)| < v$. ∎

We saw that for the pentagon (and many other graphs) $\vartheta(G) = \vartheta'(G)$. If $\mathcal{N}'_\Gamma = \emptyset$ then clearly $\vartheta(\Gamma) \neq \vartheta'(\Gamma) = \infty$. But also if $\vartheta'(\Gamma) < \infty$, $\vartheta'(\Gamma)$ may be unequal to $\vartheta(\Gamma)$. This is illustrated by the graph Γ, which is the complement of the one shown in Fig. 1. There is only one matrix in $\mathcal{N}'_\Gamma$, which is indicated by the given weights on the edges. This gives $\vartheta'(\Gamma) = 4.1198....$

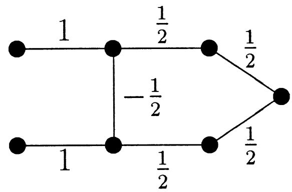

FIGURE 1

However, we easily see that Γ has chromatic number 4 and a clique of size 4, which implies that (see [7]) $\Theta(\Gamma)=\vartheta(\Gamma)=4$.

4. BOUNDS FOR BICLIQUES AND THE BICAPACITY

For a given graph Γ, let $\mathcal{M}_\Gamma$ be the class of symmetric matrices M indexed by the vertices of Γ with the property that $(M)_{i,j}=1$ if $\{i, j\}\in E$. Note that $\mathcal{N}_\Gamma$ is the subset of $\mathcal{M}_\Gamma$ whose matrices have diagonal elements equal to 1. Define (recall that $\lambda_{\text{abs}}(M)=\max\{\lambda_{\max}(M), -\lambda_{\min}(M)\}$)

$$\varphi(\Gamma)=\inf_{M\in\mathcal{M}_\Gamma}\lambda_{\text{abs}}(M).$$

(Actually, the infimum is a minimum, but we don't need it.)

Proposition 4.1. $\Phi(\Gamma)\leqslant\varphi(\Gamma)$.

Proof. Take $M\in\mathcal{M}_\Gamma$ and consider the matrices

$$\tilde{M}=\begin{bmatrix} O & M \\ M & O\end{bmatrix} \qquad \text{and} \quad \tilde{J}=\begin{bmatrix} O & J \\ J^\top & O\end{bmatrix},$$

where J has size $m\times n$ (say). If Γ has a biclique on $m+n$ vertices, then $\tilde{J}$ is a submatrix of $\tilde{M}$. The eigenvalues of $\tilde{M}$ are $\pm\lambda_i(M)$ for $i=1, ..., v$, so $\lambda_{\max}(\tilde{M})=\lambda_{\text{abs}}(M)$. We know $\lambda_{\max}(\tilde{J})=\sqrt{mn}$ and eigenvalue interlacing gives $\lambda_{\max}(\tilde{J})\leqslant\lambda_{\max}(\tilde{M})$. ∎

As in the previous section, we introduce a slightly weaker bound $\varphi'(\Gamma)$. Let $\mathcal{M}'_\Gamma$ denote the class of symmetric matrices M' indexed by the vertices of Γ with the property that $M'\underline{1}=\underline{1}$ and $(M')_{i,j}=0$ if $\{i, j\}\in E$. Define (recall that $\lambda_{\text{abs}'}(M')=\max_{i\neq 1}\{|\lambda_i(M')|\}$)

$$\varphi'(\Gamma)=\inf_{M'\in\mathcal{M}'_\Gamma}\frac{v\lambda}{1+\lambda}, \qquad \text{where} \quad \lambda=\lambda_{\text{abs}'}(M').$$

Clearly $\lambda \geqslant 0$ and $I \in \mathscr{M}'_\Gamma$, so $\mathscr{M}'_\Gamma \neq \varnothing$ and $\varphi'(\Gamma)$ is well-defined. In the next section we shall see that we may restrict ourselves to a bounded subset of $\mathscr{M}'_\Gamma$, so the infimum is a minimum. Let $\widetilde{\mathscr{M}}'_\Gamma$ be the subset of $\mathscr{M}'_\Gamma$ for which the minimum is achieved.

PROPOSITION 4.2. $\varphi(\Gamma) \leqslant \varphi'(\Gamma)$.

Proof. The proof is similar to that of Proposition 3.2. Take $M' \in \widetilde{\mathscr{M}}'_\Gamma$ and define $M = J - \frac{v}{1+\lambda} M'$. Then $M \in \mathscr{M}_\Gamma$ and $\lambda_{\text{abs}}(M) = v\lambda/(1+\lambda)$. ∎

So $\varphi'(\Gamma)$ is also an upper bound for $\Phi(\Gamma)$. In fact, this is the bound given in [4].

PROPOSITION 4.3. $\varphi'(\Gamma_1 * \Gamma_2) = \max\{v_2\varphi'(\Gamma_1), v_1\varphi'(\Gamma_2)\}$.

Proof. Take $M'_1 \in \widetilde{\mathscr{M}}'_{\Gamma_1}$ and $M'_2 \in \widetilde{\mathscr{M}}'_{\Gamma_2}$. Observe that $I \in \mathscr{M}'_\Gamma$ for any Γ, so $\lambda_{\text{abs}'}(M'_i) \leqslant 1$. Define $M' = M'_1 \otimes M'_2$. Then $M' \in \mathscr{M}'_{\Gamma_1 * \Gamma_2}$ and $\lambda_{\text{abs}'}(M') = \max\{\lambda_{\text{abs}'}(M'_1), \lambda_{\text{abs}'}(M'_2)\}$. This implies

$$\begin{aligned}\varphi'(\Gamma_1 * \Gamma_2) &\leqslant v_1 v_2 \frac{\lambda_{\text{abs}'}(M')}{1+\lambda_{\text{abs}'}(M')} \\ &= \max\left\{v_1 v_2 \frac{\lambda_{\text{abs}'}(M'_1)}{1+\lambda_{\text{abs}'}(M'_1)}, v_1 v_2 \frac{\lambda_{\text{abs}'}(M'_2)}{1+\lambda_{\text{abs}'}(M'_2)}\right\} \\ &= \max\{v_2\varphi'(\Gamma_1), v_1\varphi'(\Gamma_2)\}.\end{aligned}$$

Next take $M' \in \widetilde{\mathscr{M}}'_{\Gamma_1 * \Gamma_2}$. Consider the partition of the vertex set $V_1 \times V_2$ into v_1 subsets of size v_2 consisting of elements from $V_1 \times V_2$ with the same first coordinate. Partition the matrix M' accordingly, and define M'_1 to be the $v_1 \times v_1$ matrix consisting of the average row sums of the blocks of M'. Then $M'_1 \in \mathscr{M}'_{\Gamma_1}$ and eigenvalue interlacing (see [5]) gives $\lambda_{\text{abs}'}(M'_1) \leqslant \lambda_{\text{abs}'}(M')$. Hence

$$v_1 \frac{\lambda_{\text{abs}'}(M'_1)}{1+\lambda_{\text{abs}'}(M'_1)} \leqslant v_1 \frac{\lambda_{\text{abs}'}(M')}{1+\lambda_{\text{abs}'}(M')} = \varphi'(\Gamma_1 * \Gamma_2)/v_2.$$

Therefore $v_2\varphi'(\Gamma_1) \leqslant \varphi'(\Gamma_1 * \Gamma_2)$, and similarly $v_1\varphi'(\Gamma_2) \leqslant \varphi'(\Gamma_1 * \Gamma_2)$. ∎

Proposition 4.3 gives $\varphi'(\Gamma^k) \leqslant v^{k-1}\varphi'(\Gamma)$. Thus $\varphi'(\Gamma)$ is a bound for the bicapacity $\Phi'(\Gamma)$ of Γ:

THEOREM 4.4. $\Phi'(\Gamma) \leqslant \varphi'(\Gamma)$.

Again, if $\bar{\Gamma}$ is regular of degree k with adjacency matrix A, then $\frac{1}{k} A \in \mathscr{M}'_{\bar{\Gamma}}$. But, since the diagonal elements can be chosen freely, it is better to take a smart combination of A and I. And moreover, we don't need regularity if the Laplacian matrix is used. (If D is the diagonal matrix containing the vertex degrees of Γ, then $F = D - A$ is the Laplacian matrix. It easily follows that F is positive semi-definite with row sum $\lambda_1(F) = 0$; see, for example [1].)

COROLLARY 4.5. *Suppose F is the Laplacian matrix of Γ (Γ non-empty) and let $\lambda = \lambda_{\min'}(F)$ and $\mu = \lambda_{\max'}(F)$ ($= \lambda_{\max}(F)$). Then*

$$\Phi'(\bar{\Gamma}) \leqslant \frac{v}{2}\left(1 - \frac{\lambda}{\mu}\right).$$

Proof. Define $M' = \frac{-2}{\lambda + \mu} F + I$. Then $M' \in \mathscr{M}'_{\bar{\Gamma}}$ and $\lambda_{\text{abs}'}(M') = (\mu - \lambda)/(\mu + \lambda)$ and the result follows. ∎

Suppose Γ is the complete bipartite graph $K_{m,n}$ extended with some isolated vertices. It is straightforward to check that $\Phi(\Gamma) = \varphi(\Gamma) = \sqrt{mn}$. The complement $\bar{\Gamma}$ of Γ consists of two intersecting cliques and the eigenvalues of its Laplacian matrix are readily computed. They are 0, $v - m - n$, $v - n$, $v - m$ and v with multiplicities 1, 1, $m - 1$, $n - 1$ and $v - m - n$, respectively. So Corollary 4.5 gives $\Phi'(\Gamma) \leqslant (m + n)/2$. (In fact $\varphi'(\Gamma) = (m + n)/2$, see next section.) Thus if $m = n$ then $\Phi(\Gamma) = \Phi'(\Gamma) = m$. However, if $m \neq n$, $\Phi'(\Gamma)$ can be strictly greater than $\Phi(\Gamma)$. Indeed, let $M \subset V$ and $N \subset V$ be the parts of the $K_{m,n}$ in Γ and consider the following vertex sets of Γ^2,

$$A = \{(x, y) \mid x \in M \text{ or } y \in M\}, \qquad B = \{(x, y) \mid x \in N \text{ and } y \in N\}.$$

Then $\{A, B\}$ is a biclique in Γ^2 and $|A| \cdot |B| = n^2(2mv - m^2)$. Hence $\Phi'(\Gamma) \geqslant n\sqrt{2mv - m^2}/v$, which is larger than $\sqrt{mn}$ if and only if $m < 2v - v^2/n$. For example, if $m = 1$, $n = 3$, and $v = 5$ we obtain

$$\varphi'(\Gamma) = 2 \geqslant \Phi'(\Gamma) \geqslant \tfrac{9}{5} > \Phi(\Gamma) = \varphi(\Gamma) = \sqrt{3}.$$

This example also shows that $\varphi(\Gamma)$ is in general not an upper bound for $\Phi'(\Gamma)$. Since $2v - v^2/n$ is negative if $n > v/2$, the given example with $\Phi'(\Gamma) > \Phi(\Gamma)$ can only work if $n > v/2$ or $m > v/2$. János Körner (private communication) conjectures that $\Phi'(\Gamma) = \Phi(\Gamma)$ if both m and n are less than $v/2$. We saw that this conjecture is true if $m = n$.

5. COMPUTATION

The four considered matrix classes $\mathscr{N}_\Gamma$, $\mathscr{N}'_\Gamma$, $\mathscr{M}_\Gamma$ and $\mathscr{M}'_\Gamma$ are all convex sets. Moreover, the functions on these sets given by

$$\lambda_{\max}(N), \quad -\lambda_{\min'}(N'), \quad \lambda_{\text{abs}}(M), \qquad \text{and} \quad \lambda_{\text{abs}'}(M')$$

are convex functions. This easily follows from Rayleigh's eigenvalue inequalities. We illustrate the last one. Consider the convex combination $M' = \alpha M'_1 + \beta M'_2$ of matrices $M'_1, M'_2 \in \mathscr{M}'_\Gamma$. Let $\underline{v}$, $\underline{v} \perp \underline{1}$, be an normalised eigenvector of M' for the eigenvalue $\pm\lambda_{\mathrm{abs}'}(M')$. Then

$$\lambda_{\mathrm{abs}'}(M') = |\underline{v}^\top M' \underline{v}| \leqslant \alpha\, |\underline{v}^\top M'_1 \underline{v}| + \beta\, |\underline{v}^\top M'_2 \underline{v}| \leqslant \alpha \lambda_{\mathrm{abs}'}(M'_1) + \beta \lambda_{\mathrm{abs}'}(M'_2).$$

The other three cases go similarly. These properties make it possible to compute (or rather, approximate) the minimum value of these functions with the ellipsoid method (see Grötschel *et al.* [2, 3] or Lovász [8]). As a consequence we have the following result.

THEOREM 5.1. *The values $\vartheta(\Gamma)$, $\vartheta'(\Gamma)$, $\varphi(\Gamma)$, and $\varphi'(\Gamma)$ can be computed in polynomial time.*

Proof. For $\vartheta(\Gamma)$ this was proved by Grötschel, Lovász and Schrijver [2]. For the other three values we follow the proof for $\vartheta(\Gamma)$ given by Lovász [8, Lemma 3.2.5]. According to Theorem 2.2.15 of [8] minimization of each of the considered convex functions can be done in polynomial time, provided that the domain may be restricted to a bounded set which we know a priori.

Case $\lambda_{\mathrm{abs}}(M)$. Observe that $J \in \mathscr{M}_\Gamma$ and hence $\varphi(\Gamma) \leqslant v$. Suppose $|(M)_{i,j}| > v$ for some entry (i, j) of some $M \in \mathscr{M}_\Gamma$, then

$$\lambda^2_{\mathrm{abs}}(M) = \lambda_{\max}(M^2) \geqslant (M^2)_{i,i} = \sum_k (M)^2_{i,k} > v^2.$$

So we may restrict to matrices $M \in \mathscr{M}_\Gamma$ with $|(M)_{i,j}| \leqslant v$, which proves that $\varphi(\Gamma)$ can be computed in polynomial time.

Case $\lambda_{\mathrm{abs}'}(M')$. Put $\tilde{\lambda} = \inf_{M' \in \mathscr{M}'_\Gamma} \{\lambda_{\mathrm{abs}'}(M')\}$. Clearly $I \in \mathscr{M}'_\Gamma$, so $\tilde{\lambda} \leqslant 1$. Suppose $|(M')_{i,j}| > 2$ for some entry (i, j) of some $M' \in \mathscr{M}'_\Gamma$, then

$$\lambda^2_{\mathrm{abs}'}(M') = \lambda_{\max'}(M'^2) = \lambda_{\max}(M'^2 - J/v) \geqslant (M'^2)_{i,i} - 1/v$$

$$= \sum_k (M')^2_{i,k} - 1/v > 1.$$

So we may assume $|(M')_{i,j}| \leqslant 2$, which proves that $\tilde{\lambda}$ can be computed in polynomial time and the same is true for $\varphi'(\Gamma) = v - v/(1 + \tilde{\lambda})$.

Case $-\lambda_{\mathrm{min}'}(N')$. By Proposition 3.6, $\vartheta'(\Gamma) = \infty$ if and only if $\bar{\Gamma}$ has an isolated vertex or a component which is bipartite with parts of unequal size. This can be checked in polynomial time. Otherwise, by Proposition 3.6(ii) applied to each connected component of $\bar{\Gamma}$, there exists a $K \in \mathscr{N}'_\Gamma$

such that $|(K)_{i,j}| < v$. By use of Geršgorin's theorem (see [6, p. 346]) we find

$$\inf_{N' \in \mathcal{N}'_\Gamma} \{-\lambda_{\min'}(N')\} \leqslant -\lambda_{\min'}(K) \leqslant \lambda_{\text{abs}}(K) \leqslant \max_i \sum_j |(K)_{i,j}| < v^2.$$

Suppose $|(N')_{i,j}| = w > v^2$ for some entry (i, j) of some $N' \in \mathcal{N}'_\Gamma$. Then $\left[\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right]$ is a submatrix of N' with eigenvalues w and $-w$. So by interlacing $-\lambda_{\min}(N') \geqslant w$, hence $-\lambda_{\min'}(N') \geqslant w$. So we may take $|(N')_{i,j}| \leqslant v^2$ and therefore $\vartheta'(\Gamma)$ can be computed in polynomial time. ▮

The convexity has been used in [4] to prove the following lemma.

LEMMA 5.2. *Let $\mathcal{A}$ be an automorphism group of Γ. Then $\tilde{\mathcal{M}}'_\Gamma$ contains a matrix which is constant over each orbit of the action of $\mathcal{A}$ on $V \times V$.*

Proof. Let P_g denote the permutation matrix corresponding to $g \in \mathcal{A}$ and take $M' \in \tilde{\mathcal{M}}'_\Gamma$. Then $P_g M' P_g^\top \in \tilde{\mathcal{M}}'_\Gamma$. Define

$$\bar{M}' = \frac{1}{|\mathcal{A}|} \sum_{g \in \mathcal{A}} P_g M' P_g^\top.$$

Then clearly $\bar{M}' \in \mathcal{M}'_\Gamma$, $\bar{M}'$ is constant over $\mathcal{A}$-orbits on $V \times V$ and, since $\bar{M}'$ is a convex combination of matrices from $\tilde{\mathcal{M}}'_\Gamma$, $\lambda_{\text{abs}'}(\bar{M}') \leqslant \lambda_{\text{abs}'}(M')$. ▮

It is clear that the same result holds if $\mathcal{M}'_\Gamma$ is replaced by $\mathcal{M}_\Gamma$, $\mathcal{N}'_\Gamma$ or $\mathcal{N}_\Gamma$. Using the lemma it easily follows that $\varphi'(\Gamma) = (m+n)/2$ for the example in the previous section. It also implies that the right hand side in Corollary 4.5 equals $\varphi'(\bar{\Gamma})$ if $\bar{\Gamma}$ has an edge transitive automorphism group (Theorem 2.4 in [4]). Analogously, it follows that $\vartheta'(\bar{\Gamma})$ equals the right hand side in Corollary 3.5 if $\bar{\Gamma}$ has an edge transitive automorphism group. So these graphs satisfy $\vartheta(\bar{\Gamma}) = \vartheta'(\bar{\Gamma})$ (see Theorem 9 in [7]).

In [9] Pasechnik derives a bound for $\Phi(\Gamma)$ with semidefinite programming. Lovász's $\vartheta(\Gamma)$ also admits a description with semidefinite programming. This makes it conceivable that Pasechnik's bound coincides with $\varphi(\Gamma)$ (but it has not been worked out yet).

The complexity of the computation of $\Phi(\Gamma)$ has been an open problem for some time until recently René Peeters [10] proved that it is NP-hard.

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REFERENCES

1. R. A. Brualdi and H. J. Ryser, “Combinatorial Matrix Theory,” Cambridge Univ. Press, Cambridge, UK, 1991.
2. M. Grötschel, L. Lovász, and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, *Combinatorica* **1** (1981), 169–197.
3. M. Grötschel, L. Lovász, and A. Schrijver, “Geometric Algorithms and Combinatorial Optimization,” Springer-Verlag, Berlin/Heidelberg, 1988.
4. W. H. Haemers, Disconnected vertex sets and equidistant code pairs, *Electron. J. Combin.* **4** (1997).
5. W. H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra Appl.* **226–228** (1995), 593–616.
6. R. A. Horn and C. R. Johnson, “Matrix Analysis,” Cambridge Univ. Press, Cambridge, UK, 1985.
7. L. Lovász, On the Shannon capacity of a graph, *IEEE Trans. Inform. Theory* **25** (1979), 1–7.
8. L. Lovász, “An Algorithmic Theory of Numbers, Graphs and Convexity,” CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1986.
9. D. V. Pasechnik, Bipartite sandwiches: Bounding the size of a maximum biclique, preprint.
10. R. Peeters, The maximum edge biclique problem is NP-complete, Research Memorandum FEW789, Tilburg University, 2000.
11. C. E. Shannon, The zero-error capacity of a noisy channal, *IEEE Trans. Inform. Theory* **2** (1956), 8–19.